

Sobolev and SBV Representation Theorems for large volume limit Gibbs measures.

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Abstract

We study the limit of large volume equilibrium Gibbs measures for a rather general Hamiltonians. In particular, we study Hamiltonians which arise naturally in Nonlinear Elasticity and Hamiltonians (containing surface terms) which arises naturally in Fracture Mechanics. In both of these settings we show that an integral representation holds for the limit. Moreover, we also show a homogenization result for the Nonlinear Elasticity setting. This extends a recent result of R. Kotecký and S. Luckhaus in [8].

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1 Introduction

Recently, R. Kotecký and S. Luckhaus, have shown a remarkable result. They prove that in a fairly general setting, the limit of large volume equilibrium Gibbs measures for elasticity type Hamiltonians with clamped boundary conditions. The “zero”-temperature case was considered by R. Alicandro and M. Cicalese in [1].

Let us now briefly explain the results contained in [8].

The space of microscopic configurations consists of all $\varphi : \varepsilon\mathbb{Z}^d \rightarrow \mathbb{R}^m$. In particular, if one considers $m = d$ then this would model the elasticity situation. In this case $\varphi(x)$ can be interpreted as the displacement of the atom “positioned in x ”. When $m = 1$, this would model the random surface case, where $\varphi(x)$ can be interpreted as the height.

In order to define the Gibbs measure, we fix a configuration ψ , a set $A \subset \mathbb{R}^d$, an Hamiltonian H and a finite range interaction U . Namely there exists a set $F \subset \mathbb{Z}^d$, such that $U : \mathbb{R}^F \rightarrow \mathbb{R}$ and one

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denotes R_0 the range of the potential U i.e., $R_0 = \text{diam}(F)$. Denote by φ_F to be the restriction of φ to εF . For simplicity of notation we denote $A_\varepsilon := A \cap \varepsilon \mathbb{Z}^d$. Then they define the Hamiltonian H via

$$H_{A,\varepsilon}(\varphi) = \sum_{j \in \varepsilon \mathbb{Z}^d: \tau_j(F) \subset A} U(\varphi_{\tau_j(F)})$$

with $\tau_j(F) = \varepsilon F + j = \{i: i - j \in \varepsilon F\}$.

Finally denote by $\mathbb{1}_{A,\varepsilon,\psi}$ the indicator function of the set

$$\{\varphi \in (\mathbb{R}^m)^{A_\varepsilon} : |\varphi(i) - \psi(i)| < 1 \text{ for all } i \in S_{R_0}(\Lambda)\},$$

where

$$S_{R_0}(A) = \{x \in A_\varepsilon \mid \text{dist}(x, \varepsilon \mathbb{Z}^d \setminus A_\varepsilon) \leq \varepsilon R_0\}.$$

Then the Gibbs measure $\mu_{A,\varepsilon,\psi}(d\varphi)$ is a measure on $(\mathbb{R}^m)^{\varepsilon A}$ and is defined by

$$\mu_{A,\varepsilon,\psi}(d\varphi) = \frac{\exp\{-\beta H_{A,\varepsilon}(\varphi)\}}{Z_{A,\varepsilon,\psi}} \mathbb{1}_{A,\varepsilon,\psi}(\varphi) \prod_{i \in A_\varepsilon} d\varphi(i)$$

where $Z_{A,\varepsilon,\psi}$ is such that the above is a probability measure.

Moreover, they assume that

- (1) There exist constants $p > 0$ and $c \in (0, \infty)$ such that

$$U(\varphi_F) \geq c |\nabla \varphi(0)|^p$$

for any $\varphi \in (\mathbb{R}^m)^{\varepsilon \mathbb{Z}^d}$.

- (2) There exist $r > 1$ and $C \in (1, \infty)$ such that

$$U(s\varphi_A + (1-s)\psi_A + \eta_A) \leq C(1 + U(\varphi_A) + U(\psi_A) + \sum_{i \in \varepsilon F} |\eta(i)|^r)$$

for any $s \in [0, 1]$ and any $\varphi, \psi, \eta \in (\mathbb{R}^m)^{\varepsilon \mathbb{Z}^d}$.

With the above notation, in [8], the following theorem is proved:

Theorem 1.1. *Let U be as above with $\frac{1}{r} > \frac{1}{p} - \frac{1}{d}$. For every $u \in W^{1,p}(\Omega)$, let us define*

$$F_{\kappa,\varepsilon}(u) = -\varepsilon^d |\Omega|^{-1} \log Z_{\Omega_\varepsilon}(\mathcal{N}_{\Omega_\varepsilon,r}(u, \kappa)),$$

and

$$\begin{aligned} F_\kappa^+(u) &= \limsup_{\varepsilon \rightarrow 0} F_{\kappa,\varepsilon}(u) \\ F_\kappa^-(u) &= \liminf_{\varepsilon \rightarrow 0} F_{\kappa,\varepsilon}(u) \end{aligned}$$

Then, there exist W quasi-convex such that the following hold

- (i) $\lim_{\kappa \rightarrow 0} F_{\kappa}^{-}(u) \geq \frac{1}{|\Omega|} \int_{\Omega} W(\nabla u(x)) \, dx.$
- (ii) *If $u \in W^{1,r}(\Omega)$ then $\lim_{\kappa \rightarrow 0} F_{\kappa}^{+}(u) \leq \frac{1}{|\Omega|} \int_{\Omega} W(\nabla u(x)) \, dx.$*

They also give an explicit formula of W . Moreover, via an example, they show that W may eventually not be convex.

From the above result it is not very difficult to obtain a Large Deviation principle.

The crucial step in the proof of Theorem 1.1 is based on the possibility to approximate with partition functions on cells of a triangulation given in terms of L^r -neighbourhoods of linearizations of a minimiser of the rate functional. An important tool that allows them to impose a boundary condition on each cell of the triangulation consists in switching between the corresponding partition function $Z_{\Omega_{\varepsilon}}(\mathcal{N}_{\Omega_{\varepsilon},r}(v, \kappa))$ and the version $Z_{\Omega_{\varepsilon}}(\mathcal{N}_{\Omega_{\varepsilon},r}(v, 2\kappa) \cap \mathcal{N}_{\Omega_{\varepsilon},R_0,\infty}(Z))$ with an additional soft clamp $|\varphi(i) - \psi(i)| < 1$ enforced in the boundary strip of the width $R_0 > \text{diam}(A)$ with $Z \in \mathcal{N}_{\Omega_{\varepsilon},r}(v, \kappa)$ arbitrarily chosen.

We improve their result in the following manner:

- (i) We consider Hamiltonians, where the interaction is *not* of finite range and is dependent¹ on the scale ε and the position x . We are also able to give an homogenisation result.
- (ii) By considering a different version of the interpolation argument we are able to consider “hard” boundary condition instead of the clamped ones. In our opinion, this type of boundary conditions are more in line with the standard theory of Statistical Mechanics.
- (iii) We simplify some of the arguments by relying on the representation formulas, hence avoiding triangulation arguments.
- (iv) We are able to consider more general potentials, which “relax” in SBV.

2 Sobolev Representation Theorems

2.1 Preliminary results

Let Ω be an open set. We denote by $\mathcal{A}(\Omega)$ the family of all open sets contained in Ω . We now recall a well-known result in measure theory due to E. De Giorgi and G. Letta. The proof can be found in [2].

Theorem 2.1. *Let X be a metric space and let us denote by \mathcal{A} its open sets. Let $\mu : \mathcal{A} \rightarrow [0, \infty]$ be an increasing set function such that*

- (DL1) $\mu(\emptyset) = 0;$
- (DL2) $A, B \in \mathcal{A}$ then $\mu(A \cup B) \leq \mu(A) + \mu(B);$
- (DL3) $A, B \in \mathcal{A}$, such that $A \cap B = \emptyset$ then $\mu(A \cap B) \geq \mu(A) + \mu(B)$
- (DL4) $\mu(A) = \sup \{\mu(B) : B \subseteq A\}.$ Then, the extension of μ to every $C \subset X$ given by

$$\mu(C) = \inf \{\mu(A) : A \in \mathcal{A}, A \supset C\}$$

¹for the precise definition see the next section

is an outer measure. In particular the restriction of μ to the Borel σ -algebra is a positive measure.

We recall the well-known integral representation formulas (see [6]).

Theorem 2.2. *Let $1 \leq p < \infty$ and let $F : W^{1,p} \times \mathcal{A}(\Omega) \rightarrow [0, +\infty]$ be a functional satisfying the following conditions:*

- (i) (locality) F is local, i.e. $F(u, A) = F(v, A)$ if $u = v$ a.e. on $A \in \mathcal{A}(\Omega)$;
- (ii) (measure property) for all $u \in W^{1,p}$ the set function $F(u, \cdot)$ is the restriction of a Borel measure to $\mathcal{A}(\Omega)$;
- (iii) (growth condition) there exists $c > 0$ and $a \in L^1(\Omega)$ such that

$$F(u, A) \leq c \int_A (a(x) + |Du|^p) \, dx$$

for all $u \in W^{1,p}$ and $A \in \mathcal{A}(\Omega)$;

- (iv) (translation invariance in u) $F(u + z, A) = F(u, A)$ for all $z \in \mathbb{R}^d$, $u \in W^{1,p}$ and $A \in \mathcal{A}(\Omega)$;
- (v) (lower semicontinuity) for all $A \in \mathcal{A}(\Omega)$ $F(\cdot, A)$ is sequentially lower semicontinuous with respect to the weak convergence in $W^{1,p}$.

Then there exists a Carathéodory function $f : \Omega \times \mathbb{M}^{d \times N} \rightarrow [0, +\infty)$ satisfying the growth condition

$$0 \leq f(x, M) \leq c(a(x) + |M|^p)$$

for all $x \in \Omega$ and $M \in \mathbb{M}^{d \times N}$, such that

$$F(u, A) = \int_A f(x, Du(x)) \, dx$$

for all $u \in W^{1,p}$ and $A \in \mathcal{A}(\Omega)$.

If in addition it holds

- (vi) (translation invariance in x)

$$F(Mx, B(y, \varrho)) = F(Mx, B(z, \varrho))$$

for all $M \in \mathbb{M}^{d \times N}$, $y, z \in \Omega$, and $\varrho > 0$ such that $B(y, \varrho) \cup B(z, \varrho) \subset \Omega$, then f does not depend on x .

2.2 Hypothesis and Main Theorem

For any $u \in L^1_{\text{loc}}(\mathbb{R}^d, \mathbb{R}^m)$, let $X_{u,\varepsilon} : \mathbb{Z}^d \rightarrow \mathbb{R}^m$ and $\varphi : \varepsilon\mathbb{Z}^d \rightarrow \mathbb{R}^m$ be defined by

$$\begin{aligned} X_{u,\varepsilon}(i) &= \frac{1}{\varepsilon} \int_{\varepsilon i + Q(\varepsilon)} u(y) \, dy \\ \varphi_{u,\varepsilon}(\varepsilon i) &= \frac{1}{\varepsilon} \int_{\varepsilon i + Q(\varepsilon)} u(y) \, dy \end{aligned} \tag{1}$$

for any $i \in \mathbb{Z}^d$. Here, $Q(\varepsilon) = [-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}]^d$ and \bar{f} denotes the mean value, i.e., for every $f \in L^1(\mathbb{R}^d)$

$$\bar{f}_A = \frac{1}{|A|} \int_A f(x) dx$$

Let $u \in W^{1,p}(\mathbb{R}^d)$, A is an open set and $p \geq 1$. Then it is not difficult to prove that

$$\lim_{\varepsilon \downarrow 0} \sum_{x \in A_\varepsilon} \varepsilon^d |\nabla \varphi_u(x)|^p = \int_A |\nabla u|^p. \quad (2)$$

On the other hand, let

$$\Pi_\varepsilon : (\mathbb{R}^m)_0^{\mathbb{Z}^d} \rightarrow W^{1,p}(\mathbb{R}^d) \quad (3)$$

be a canonical interpolation $X \rightarrow v$ such that $v(\varepsilon i) = \varepsilon \varphi(\varepsilon i)$ for any $i \in \mathbb{Z}^d$. Here, $(\mathbb{R}^m)_0^{\mathbb{Z}^d}$ is the set of functions $X : \mathbb{Z}^d \rightarrow \mathbb{R}^m$ with finite support. To fix ideas, we can consider a triangulation of \mathbb{Z}^d into simplexes with vertices in $\varepsilon \mathbb{Z}^d$, and choose v on each simplex as the linear interpolation of the values $\varepsilon \varphi(\varepsilon i)$ on the vertices εi .

Let Ω be an open set with regular boundary. We denote by $\Omega_\varepsilon = \varepsilon \mathbb{Z}^d \cap \Omega$ and by $\mathcal{A}(\Omega)$ the set of all open sets contained in Ω with regular boundary. For every set $A \in \mathcal{A}(\Omega)$, we define

$$R_\varepsilon^\xi(A) := \{\alpha \in \varepsilon \mathbb{Z}^d \mid [\alpha, \alpha + \varepsilon \xi] \subset A\},$$

where by $[x, y]$ we mean the segment connecting x and y , i.e., $\{\lambda x + (1 - \lambda)y : \lambda \in [0, 1]\}$.

The Hamilton H is defined by

$$H(\varphi, \varepsilon) := \sum_{\xi \in \mathbb{Z}^d} \sum_{x \in R_\varepsilon^\xi(\Omega)} f_{\xi, \varepsilon}(x, \nabla_\xi \varphi),$$

where $\xi \in \mathbb{Z}^d$, and

$$\nabla_\xi \varphi(x) := \frac{\varphi(x + \varepsilon \xi) - \varphi(x)}{|\xi|}.$$

We also define the Hamiltonian taking into account the contribution from the boundary as

$$H_\infty(\varphi, A, \varepsilon) := \sum_{\xi \in \mathbb{Z}^d} \sum_{x \in A_\varepsilon} f_{\xi, \varepsilon}(x, \nabla_\xi \varphi(x)).$$

The functions $f_{\xi, \varepsilon}$ will be specified later.

In order to apply the representation formulas, we shall need to localize. For this reason, for every $\varepsilon > 0$ and $A \subset \Omega$ open, set we introduce

$$H(\varphi, A, \varepsilon) := \sum_{\xi \in \mathbb{Z}^d} \sum_{x \in R_\varepsilon^\xi(A)} f_{\xi, \varepsilon}(x, \nabla_\xi \varphi(x)).$$

For simplicity of notation, we will also denote

$$H^\xi(\varphi, A, \varepsilon) := \sum_{x \in R_\varepsilon^\xi(A)} f_{\xi, \varepsilon}(x, \nabla_\xi \varphi(x)).$$

Moreover, let $\{e_1, \dots, e_d\}$ be the standard basis of \mathbb{R}^d . In this section, the functions $f_{\xi, \varepsilon}$ will satisfy the followings

(C1) $f_{\xi, \varepsilon} > 0$;

(C2) there exist constants C_ξ such that

$$f_{\xi, \varepsilon}(x, s + t) \leq f_{\xi, \varepsilon}(x, s) + C_\xi(|t|^p + 1);$$

where the constants C_ξ satisfy

$$\sum_{\xi \in \mathbb{Z}^d} C_\xi < +\infty;$$

(C3) there exists a constant C such that

$$f_{e_i, \varepsilon}(x, t) \geq C \max(|t|^p - 1, 0).$$

For every $A \in \mathcal{A}(\Omega)$, we define the free-energy as

$$\begin{aligned} F(u, A, \kappa, \varepsilon) &:= -\frac{\varepsilon^d}{|A|} \log \int_{\mathcal{V}(u, A, \kappa, \varepsilon)} \exp \left(-H(\varphi, A, \varepsilon) \right) d\varphi \\ F_\infty(u, A, \kappa, \varepsilon) &:= -\varepsilon^d \log \int_{\mathcal{V}_\infty(u, A, \kappa, \varepsilon)} \exp \left(-H_\infty(\varphi, A, \varepsilon) \right) d\varphi, \end{aligned} \tag{4}$$

where

$$\begin{aligned} \mathcal{V}(u, A, \kappa, \varepsilon) &= \left\{ \varphi : A_\varepsilon \rightarrow \mathbb{R}^m \mid \frac{\varepsilon^d}{|A|^d} \sum_{x \in A_\varepsilon} |u - \varepsilon \varphi|^p \leq \kappa^p \right\} \\ \mathcal{V}_\infty(u, A, \kappa, \varepsilon) &= \left\{ \varphi : \varepsilon \mathbb{Z}^d \rightarrow \mathbb{R}^m \mid \frac{\varepsilon^d}{|A|^d} \sum_{x \in A_\varepsilon} |u - \varepsilon \varphi|^p \leq \kappa^p, \text{ and } \varphi(x) = \varphi_{u, \varepsilon}(x) \ \forall x \notin A_\varepsilon \right\}, \end{aligned}$$

where $\varphi_{u, \varepsilon}$ is defined in (1).

Let us introduce the following notations:

$$\begin{aligned}
F'(u, A, \kappa) &:= \liminf_{\varepsilon \downarrow 0} F(u, A, \kappa, \varepsilon) \\
F''(u, A, \kappa) &:= \limsup_{\varepsilon \downarrow 0} F(u, A, \kappa, \varepsilon) \\
F'(u, A) &:= \lim_{\kappa \downarrow 0} \liminf_{\varepsilon \downarrow 0} F(u, A, \kappa, \varepsilon) = \lim_{\kappa \downarrow 0} F'(u, A, \kappa) \\
F''(u, A) &:= \lim_{\kappa \downarrow 0} \limsup_{\varepsilon \downarrow 0} F(u, A, \kappa, \varepsilon) = \lim_{\kappa \downarrow 0} F''(u, A, \kappa) \\
F'_\infty(u, A, \kappa) &:= \liminf_{\varepsilon \downarrow 0} F_\infty(u, A, \kappa, \varepsilon) \\
F''_\infty(u, A, \kappa) &:= \limsup_{\varepsilon \downarrow 0} F_\infty(u, A, \kappa, \varepsilon) \\
F'_\infty(u, A) &:= \lim_{\kappa \downarrow 0} \liminf_{\varepsilon \downarrow 0} F_\infty(u, A, \kappa, \varepsilon) = \lim_{\kappa \downarrow 0} F'_\infty(u, A, \kappa) \\
F''_\infty(u, A) &:= \lim_{\kappa \downarrow 0} \limsup_{\varepsilon \downarrow 0} F_\infty(u, A, \kappa, \varepsilon) = \lim_{\kappa \downarrow 0} F''_\infty(u, A, \kappa)
\end{aligned} \tag{5}$$

One of the main steps will be to show that $F'_\infty = F'$ and that $F''_\infty = F''$. The basic intuition behind is the so called interpolation lemma, which is well-known in the Γ -convergence community. Very informally, what it says that if one imposes “closeness” in $L^p(A)$ to some regular function u , then one can also impose the boundary condition by “paying a very small price in energy”. More precisely, given a sequence $\{v_n\}$ such that $v_n \rightarrow u$ in $L^p(A)$, where A is an open set, then there exists a sequence $\{\tilde{v}_n\}$, such that $\tilde{v}_n \rightarrow u$ in $L^p(A)$, $\tilde{v}_n|_{\partial\Omega} = u|_{\partial\Omega}$ and such that

$$\liminf_n \int_A |\nabla \tilde{v}_n|^2 \leq \liminf_n \int_A |\nabla v_n|^2.$$

Remark 2.3. (i) *The functional $F(u, A, \kappa, \varepsilon)$ is monotonically decreasing in $\delta, \kappa > 0$, i.e.*

$$F(u, A, \kappa, \varepsilon) \leq F(u, A, \kappa + \delta, \varepsilon).$$

This justifies the outer limit in the formulas of (5). Moreover, the outer limit in the formulas in (5) can be substituted with the supremum i.e.,

$$\begin{aligned}
F'(u, A) &:= \sup_{\kappa > 0} \liminf_{\varepsilon \downarrow 0} F(u, A, \kappa, \varepsilon) = \sup_{\kappa > 0} F'(u, A, \kappa), \\
F''(u, A) &:= \sup_{\kappa > 0} \limsup_{\varepsilon \downarrow 0} F(u, A, \kappa, \varepsilon) = \sup_{\kappa > 0} F''(u, A, \kappa).
\end{aligned}$$

(ii) *Let A, B be two open sets such that $A \cap B = \emptyset$, then from the definitions it is not difficult to prove that*

$$F'(u, A) + F'(u, B) = F'(u, A \cup B) \quad \text{and} \quad F''(u, A) + F''(u, B) = F''(u, A \cup B).$$

(iii) *Whenever the function u is linear and the functions $f_{\xi, \varepsilon}$ do not depend on ε and the space variable x , it is well-known that $F' = F''$. In Theorem 2.18, we are going to prove a more general result, which contains as a particular case the previous claim.*

Proposition 2.4. *The maps F', F'' are lower semicontinuous with respect to the $L^p(A)$ convergence. Moreover, there exists a sequence $\{\varepsilon_n\}$ such that*

$$F'_{\{\varepsilon_n\}}(u) = F''_{\{\varepsilon_n\}}(u), \quad (6)$$

where

$$F'_{\{\varepsilon_n\}}(u) := \lim_{\kappa \downarrow 0} \liminf_{n \rightarrow \infty} F(u, A, \kappa, \varepsilon_n) \quad \text{and} \quad F''_{\{\varepsilon_n\}}(u) := \lim_{\kappa \downarrow 0} \limsup_{n \rightarrow \infty} F(u, A, \kappa, \varepsilon_n).$$

Proof. Let us recall the notations

$$F'_{\{\varepsilon_k\}}(u, A, \kappa) = \liminf_{n \rightarrow +\infty} F(u, A, \kappa, \varepsilon_n) \quad \text{and} \quad F''_{\{\varepsilon_k\}}(u, A, \kappa) = \limsup_{n \rightarrow +\infty} F(u, A, \kappa, \varepsilon_n).$$

Using $F(v, A, \kappa, \varepsilon) \geq F(u, A, \kappa + \delta, \varepsilon)$ where $\|u - v\|_{L^p(A)} < \delta$, one has that

$$F'(v, A, \kappa) = \liminf_{n \rightarrow \infty} F(u, A, \kappa, \varepsilon_n) \geq \liminf_{n \rightarrow \infty} F(v, A, \kappa + \delta, \varepsilon_n) = F'(u, A, \kappa + \delta).$$

Thus,

$$\liminf_{v \rightarrow u} \sup_{\kappa > 0} F'(u, A, \kappa) \geq \sup_{\kappa > 0} F'(u, A, \kappa + \delta)$$

and finally passing also to the supremum in δ one has that F' is lower semicontinuous. The statement for F'' follows in a similar fashion.

Fix \mathcal{D} a countable dense set in $L^p(A)$ and let \mathcal{U} be the set of all balls centered in the elements of \mathcal{D} with radii in $[0, 1] \cap \mathbb{Q}$. Let us enumerate the balls in \mathcal{U} , namely $\mathcal{U} := \{B_i : i \in \mathbb{N}\}$.

Let $u_1 \in B_1$ be such that

$$F'(u_1, A) \leq \inf_{u \in B_1} F'(u) + \text{diam}(B_1).$$

Let $\{\varepsilon_n^{(1)}\}$ be the sequence such that

$$F'(u_1, A) = F''(u_1, A) = \lim_{\kappa \downarrow 0} \lim_{n \rightarrow \infty} F(u_1, A, \kappa, \varepsilon_n^{(1)}).$$

In a similar way as for B_1 , let $u_2 \in B_2$ be such that

$$F'_{\{\varepsilon_n^{(1)}\}}(u_2, A) \leq \inf_{u \in B_2} F'_{\{\varepsilon_n^{(1)}\}}(u) + \text{diam}(B_2).$$

Moreover, let $\{\varepsilon_n^{(2)}\} \subset \{\varepsilon_n^{(1)}\}$ be such that

$$F'(u_2, A) = \lim_{\kappa \downarrow 0} \lim_{n \rightarrow \infty} F(u, A, \kappa, \varepsilon_n^{(2)}).$$

By an induction procedure it is possible to produce a subsequence $\{\varepsilon_n^{(k+1)}\} \subset \{\varepsilon_n^{(k)}\}$ such that

$$F'(u_k, A) = \lim_{\kappa \downarrow 0} \lim_{n \rightarrow \infty} F(u_k, A, \kappa, \varepsilon_n^{(k)}),$$

where u_k is chosen such that

$$F'_{\{\varepsilon_n^{(k+1)}\}}(u_{k+1}, A) \leq \inf_{B_{k+1}} F'_{\{\varepsilon_n^{(k)}\}} + \text{diam}(B_{k+1}).$$

By a diagonal argument it is possible to chose a single sequence $\{\varepsilon_k\}$, such that all the above are satisfied. Because the second claim of the Proposition 2.4 consists in showing (6) for a particular sequence, one can assume without loss of generality that it satisfies the above relations.

Let us now show that $F'_{\{\varepsilon_n\}} = F''_{\{\varepsilon_n\}}$. From the definitions it is trivial that $F'_{\{\varepsilon_n\}} \leq F''_{\{\varepsilon_n\}}$. Let us now show the opposite inequality. Fix u . For every i such that $u \in B_i$ we have that²

$$F'_{\{\varepsilon_n\}}(u, A) + \text{diam}(B_i) \geq F'_{\{\varepsilon_n\}}(u_i, A) = F''_{\{\varepsilon_n\}}(u_i, A).$$

Passing to the limit for $i \rightarrow \infty$, using the lower semicontinuity of $F''_{\{\varepsilon_n\}}$, and the arbitrariness of $\text{diam}(B_i)$, we have the desired result. \square

Fix Ω an open set, $\varepsilon > 0$ and $u \in W^{1,p}(\mathbb{R}^d)$ and let $\varphi_{u,\varepsilon}$ be defined by in (1).

We are now able to write the main result in this section:

Theorem 2.5. *Assume the above hypothesis. Then for every infinitesimal sequence (ε_n) there exists a subsequence (ε_{n_k}) and there exists a function $W : \Omega \times \mathbb{R}^{d \times m} \rightarrow \mathbb{R}$ (depending on $\{\varepsilon_{n_k}\}$) such that*

$$F'_{\{\varepsilon_{n_k}\}}(u, A) = F''_{\{\varepsilon_{n_k}\}}(u, A) = \int_A W(x, \nabla u) \, dx. \quad (7)$$

2.3 Proofs

The next technical lemma asserts that finite difference quotients along any direction can be controlled by finite difference quotients along the coordinate directions:

Lemma 2.6 ([1, Lemma 3.6]). *Let $A \in \mathcal{A}(\Omega)$ and set $A_\varepsilon = \{x \in A : \text{dist}(x, \partial A) > 2\sqrt{d}\varepsilon\}$. Then for any $\xi \in \mathbb{Z}^d$ and $\varphi : A_\varepsilon \rightarrow \mathbb{R}^m$, it holds*

$$\sum_{x \in R_\varepsilon^\xi(A)} \left| \frac{\varphi(x + \varepsilon\xi) - \varphi(x)}{|\xi|} \right|^p \leq C \sum_{i=1}^d \sum_{x \in R_\varepsilon^{e_i}(A)} |\nabla_i \varphi(x)|^p, \quad (8)$$

where the constant C is independent of ξ .

The following lemma is a simple modification of [8, Lemma A.1]:

Lemma 2.7. *Let $g : \mathbb{R} \rightarrow [0, +\infty]$ be such that*

$$\int_{\mathbb{R}} \exp(-g(t)) \, dt =: c.$$

Then there exists ε_0 such that for every $\varepsilon \leq \varepsilon_0$, it holds

$$\int_{\mathcal{V}(0, A, \kappa, \varepsilon)} \exp\left(- \sum_{x, x+\varepsilon e_i \in A_\varepsilon} g(|\nabla \varphi(x)|)\right) \leq C |A_\varepsilon| \kappa \varepsilon^{-1} \exp\left(|A_\varepsilon| \log(c)\right) \, d\varphi.$$

²by the above construction

Proof. The proof is a simple modification of [8, Lemma A.1], for this reason we do not give a proof. \square

Let G^λ be the free-energy (see (4) for the definition) induced by the Hamiltonian

$$\tilde{H}^\lambda(\varphi, A, \varepsilon) := \lambda \sum_{i=1}^d \sum_{x \in R_\varepsilon^{e_i}(A)} |\nabla_i \varphi|^p.$$

Lemma 2.8. *There exist constants C_λ, D_λ , such that it holds*

$$C_\lambda \leq \liminf_{\varepsilon \downarrow 0} G^\lambda(0, A, \kappa, \varepsilon) \leq \limsup_{\varepsilon \downarrow 0} G^\lambda(0, A, \kappa, \varepsilon) \leq D_\lambda$$

Proof. Let us prove now the upper bound, namely

$$G^\lambda(0, A, \kappa, \varepsilon) \leq D_\lambda. \quad (9)$$

Let us observe that

$$\tilde{H}^\lambda(\varphi, A, \varepsilon) \leq 2^{p-1} d \cdot \lambda \sum_{x \in A_\varepsilon} |\varphi(x)|^p, \quad (10)$$

hence

$$\int_{\mathcal{V}(0, A, \kappa, \varepsilon)} \exp\left(-\tilde{H}^\lambda(\varphi, A, \varepsilon)\right) \geq \int_{\{\varphi: |\varepsilon \varphi| \leq \kappa\}} \exp\left(-\sum_{x \in A_\varepsilon} |\varphi(x)|^p\right).$$

Thus, by using the Fubini Theorem, we have that

$$\int_{\mathcal{V}(0, A, \kappa, \varepsilon)} \exp\left(-\tilde{H}^\lambda(\varphi, A, \varepsilon)\right) \geq \exp\left(-\varepsilon^{-d} D_{\varepsilon, \varepsilon}\right),$$

where

$$D_{\varepsilon, \lambda} := -\log \int_{|t| \leq \kappa/\varepsilon} \exp(-t^p).$$

Using the definition of the free-energy, and the fact that $D_{\varepsilon, \lambda} \rightarrow -\log \int_{\mathbb{R}} \exp(-t^p)$, we have the first part of the claim.

The second inequality is implied by Lemma 2.7. \square

Lemma 2.9. *Let $\{f_{\xi, \varepsilon}\}$ satisfy our hypothesis. Then there exists a constant D such that for every $\kappa < 1$, one has that*

$$\exp\left(-\varepsilon^{-d} F(u, A, \kappa, \varepsilon)\right) \leq \exp\left(D \varepsilon^{-d} + D \sum_{i=1}^d \sum_{x \in R_\varepsilon^{e_i}(A)} |\nabla_{e_i} \varphi_{u, \varepsilon}(x)|^p\right), \quad (11)$$

where $\varphi_{u, \varepsilon}$.

Proof. Given that $\|b - a\|^p \geq 2^{1-p}\|a\|^p - \|b\|^p$ one has that there exists a constant C_1 such that

$$H(\varphi, A, \varepsilon) \geq C_1 \sum_{i=1}^d \sum_{x \in R_\varepsilon^{e_i}(A)} |\nabla_{e_i} \varphi(x)|^p \geq C_1 \sum_{i=1}^d \sum_{x \in R_\varepsilon^{e_i}(A)} |\nabla \psi|^p - C_1 \sum_{i=1}^d \sum_{x \in R_\varepsilon^{e_i}(A)} |\nabla_{e_i}(\varphi_{u,\varepsilon})(x)|^p,$$

where $\psi = \varphi - \varphi_{u,\varepsilon}$ and $\varphi_{u,\varepsilon}$ is defined in (1). Hence, the estimate (11) reduces to prove that there exists a constant D such that

$$\int_{\mathcal{V}(0,A,\kappa,\varepsilon)} \exp \left(-C \sum_{i=1}^d \sum_{x \in R_\varepsilon^{e_i}(A)} |\nabla_{e_i} \varphi|^p \right) \leq \exp \left(D\varepsilon^{-d} \right).$$

The above inequality was proved in Lemma 2.8. □

Remark 2.10. A simple consequence of the reasoning done in Lemma 2.9, is that there exists a constant C such that

$$A \mapsto F'(u, A) + C(|\nabla u|_{L^p(A)} + 1) \quad A \mapsto F''(u, A) + C(|\nabla u|_{L^p(A)} + 1)$$

are monotone with respect to the inclusion relation i.e., for every $A \subset B$ it holds that

$$F'(u, A) + C(|\nabla u|_{L^p(A)} + 1) \leq F'(u, B) + C(|\nabla u|_{L^p(B)} + 1).$$

Lemma 2.11. Let $f_{\xi,\varepsilon}$ satisfy our hypothesis and let A be an open set. Then there exists a constant $D > 0$, such that

$$\exp \left(-\varepsilon^{-d} F(u, A, \kappa, \varepsilon) \right) \geq \exp \left(-D\varepsilon^{-d} - \sum_{i=1}^d \sum_{x \in R_\varepsilon^{e_i}(A)} |\nabla_{e_i} \varphi_{u,\varepsilon}(x)| \right), \quad (12)$$

where $\varphi_{u,\varepsilon}$ is defined in (1).

Proof. Using Lemma 2.6, one has that there exists a constant C such that

$$H(\varphi, A, \varepsilon) \leq C \sum_{i=1}^d \sum_{x \in R_\varepsilon^{e_i}(A)} |\nabla_{e_i} \varphi(x)|^p$$

Given that $\|a + b\|^p \leq 2^{p-1}\|a\|^p + 2^{p-1}\|b\|^p$, there exist a constant C_1 such that

$$H(\varphi, A, \varepsilon) \leq C_1 \sum_{i=1}^d \sum_{x \in R_\varepsilon^{e_i}(A)} (|\nabla_{e_i} \varphi_{u,\varepsilon}|^p + 1) + \sum_{i=1}^d \sum_{x \in R_\varepsilon^{e_i}(A)} |\nabla_{e_i} \psi(x)|^p,$$

where $\psi = \varphi - \varphi_{u,\varepsilon}$. Hence, the estimate (12) reduces to prove that there exists a constant D such that

$$\int_{\mathcal{V}(0,A,\kappa,\varepsilon)} \exp \left(-C \sum_{i=1}^d \sum_{x \in R_\varepsilon^{e_i}(A)} |\nabla_{e_i} \varphi|^p \right) \leq \exp \left(D\varepsilon^{-d} \right).$$

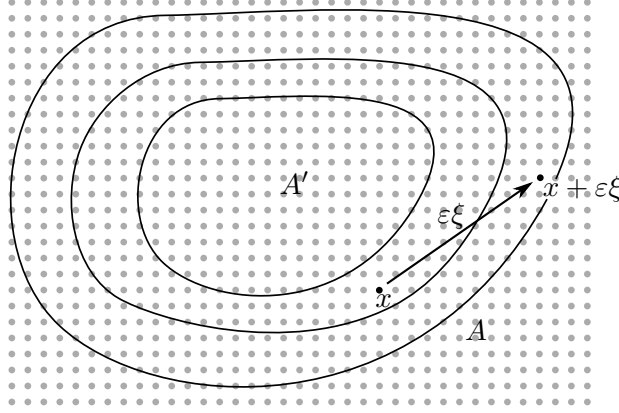


Figure 1

The above inequality was proved in Lemma 2.8. □

Lemma 2.12 (exponential tightness). *Let A be an open set and $K \geq 0$. Denote by*

$$\mathcal{M}_K := \left\{ \varphi : H(\varphi, A, \varepsilon) \geq K\varepsilon^{-d}|A| \right\}.$$

Then there exist constants D, K_0, ε_0 such that for every $K \geq K_0$, $\varepsilon \leq \varepsilon_0$ and $u \in L^p(A)$ it holds

$$\int_{\mathcal{M}_K \cap \mathcal{V}(u, A, \kappa)} \exp(-H(\varphi, A, \varepsilon)) \leq \exp\left(-\frac{1}{2}K\varepsilon^{-d} + D\varepsilon^{-d} - D \sum_{i=1}^d \sum_{x \in R_\varepsilon^i(A)} |\nabla_{e_i} \varphi_u|^p\right)$$

Proof. For every $\varphi \in \mathcal{M}_K$ it holds

$$H(\varphi, A, \varepsilon) \geq K/2\varepsilon^{-d} + \frac{1}{2}H(\varphi, A, \varepsilon).$$

Hence, by using Lemma 2.9, we have the desired result. □

We will now proceed to prove the hypothesis of Theorem 2.2.

Even though in the next two lemmas a very similar reasoning is used, they cannot be derived one from the other.

Lemma 2.13 (regularity). *Let $f_{\xi, \varepsilon}$ satisfy the usual hypothesis and $u \in W^{1,p}(\Omega)$. Then*

$$\sup_{A' \in \mathcal{A}} F''(u, A') = F''(u, A).$$

Proof. Let us fix $A' \Subset A$ and $N \in \mathbb{N}$ (to be chosen later). Let $\delta = \text{dist}(A', A^C)$, and let $0 < t_1, \dots, t_N \leq \delta$ such that $t_{i+1} - t_i > \frac{\delta}{2N}$. Without loss of generality, we may assume that there exists no $x \in A_\varepsilon$ such that $\text{dist}(x, A^C) = t_i$. For every i , we define

$$A_i := \{x \in A_\varepsilon : \text{dist}(x, A^C) \geq t_i\}$$

and

$$S_i^{\xi, \varepsilon} := \{x \in A_i : x + \varepsilon \xi \in A \setminus A_i\}.$$

With the above definitions it holds

$$R_\varepsilon^\xi(A) = R_\varepsilon^\xi(A') \cup R_\varepsilon^\xi(A \setminus \bar{A}') \cup S_i^{\varepsilon, \xi},$$

thus

$$H^\xi(\varphi, A, \varepsilon) \leq H^\xi(\varphi, A \setminus \bar{A}_i, \varepsilon) + H^\xi(\varphi, A_i, \varepsilon) + \sum_{x \in S_i^{\xi, \varepsilon}} f_{\xi, \varepsilon}(\nabla \varphi(x)).$$

Hence, by using hypothesis (C2) one has that,

$$H(\varphi, A, \varepsilon) \leq H(\varphi, A_i, \varepsilon) + H(\varphi, A \setminus A_i, \varepsilon) + \sum_{\xi \in \mathbb{Z}^d} \sum_{x \in S_i^{\xi, \varepsilon}} C_\xi (|\nabla_\xi \varphi(x)|^p + 1).$$

Let us now estimate the last term in the previous inequality.

We separate the sum into two terms

$$\sum_{\xi \in \mathbb{Z}^d} \sum_{x \in S_i^{\xi, \varepsilon}} |\nabla_\xi \varphi(x)|^p = \sum_{|\xi| \leq M} \sum_{x \in S_i^{\xi, \varepsilon}} |\nabla_\xi \varphi(x)|^p + \sum_{|\xi| > M} \sum_{x \in S_i^{\xi, \varepsilon}} |\nabla_\xi \varphi(x)|^p, \quad (13)$$

where $M \in \mathbb{N}$. From hypothesis (C2) and by taking M sufficiently large, we may also assume without loss of generality that

$$\sum_{|\xi| \geq M} C_\xi \leq \delta_1,$$

hence using Lemma 2.6,

$$\sum_{|\xi| \geq M} \sum_{x \in S_i^{\xi, \varepsilon}} |\nabla_\xi \varphi(x)|^p \leq C \delta_1 \sum_{k=1}^d \sum_{x \in R_\varepsilon^{e_k}(A)} |\nabla_{e_k} \varphi(x)|^p \leq \tilde{C} \delta_1 H(\varphi, A, \varepsilon),$$

where in the last inequality we have used hypothesis (C3).

Let $|\xi| < M$. If $\varepsilon M N \leq 2\delta$, then

$$S_i^{\xi, \varepsilon} \cap S_j^{\xi, \varepsilon} = \emptyset \quad \text{whenever } |i - j| \geq 2.$$

Without loss of generality we may assume the above condition as $\varepsilon \rightarrow 0$.

Given that

$$\frac{1}{N-2} \sum_{i=1}^{N-2} \sum_{|\xi| < M} \sum_{x \in S_i^{\varepsilon, \xi}} |\nabla \varphi(x)|^p \leq 2CH(\varphi, A, \varepsilon),$$

there exist $0 < i \leq N-2$ such that

$$\sum_{|\xi| < M} \sum_{x \in S_i^{\varepsilon, \xi}} |\nabla_{\xi} \varphi|^p < \frac{2}{N-2} H(\varphi, A, \varepsilon). \quad (14)$$

Let us denote by \mathcal{N}_i the set of all $\varphi \in \mathcal{V}(u, A, \kappa, \varepsilon)$ such that (14) holds for the first time, namely for every $j \leq i$

$$\sum_{|\xi| < M} \sum_{x \in S_i^{\varepsilon, \xi}} |\nabla_{\xi} \varphi|^p \geq \frac{2}{N-2} H(\varphi, A, \varepsilon) \quad (15)$$

On one side, one has that

$$\int_{\mathcal{V}(u, A, \kappa, \varepsilon)} \exp(-H(\varphi, A, \varepsilon)) \leq \sum_{i=1}^N \int_{\mathcal{N}_i} \exp(-H(\varphi, A_i, \varepsilon) - H(\varphi, A \setminus \bar{A}_i, \varepsilon)). \quad (16)$$

On the other side, one has that

$$\int_{\mathcal{V}(u, A, \kappa, \varepsilon)} \exp(-H(\varphi, A, \varepsilon)) \geq \sum_{i=1}^N \int_{\mathcal{N}_i^K} \exp(-H(\varphi, A, \varepsilon)), \quad (17)$$

where $\mathcal{N}_i^K := \mathcal{N}_i \setminus \mathcal{M}_K$. By using (15), one has that for every $\varphi \in \mathcal{N}_i^K$ it holds

$$H(\varphi, A, \varepsilon) \leq H(\varphi, A_i) + H(\varphi, A \setminus \bar{A}_i) + \frac{K\varepsilon^{-d}}{N-2},$$

and for every φ it holds

$$H(\varphi, A, \varepsilon) \geq H(\varphi, A_i, \varepsilon) + H(\varphi, A \setminus \bar{A}_i, \varepsilon). \quad (18)$$

Hence,

$$\int_{\mathcal{V}(u, A, \kappa, \varepsilon)} \exp(-H(\varphi, A, \varepsilon)) \geq \sum_{i=1}^N \int_{\mathcal{N}_i^K} \exp\left(-H(\varphi, A_i, \varepsilon) - H(\varphi, A \setminus \bar{A}_i, \varepsilon) - \frac{K\varepsilon^{-d}}{N-2}\right).$$

By using Lemma 2.12, i.e., the fact that there exist K_0 , ε_0 and D such that for every $K > K_0$ and $\varepsilon \leq \varepsilon_0$ one has that

$$\int_{\mathcal{M}_K \cap \mathcal{V}(u, A, \kappa, \varepsilon)} \exp(-H(\varphi, A, \varepsilon)) \leq \exp\left(-\frac{1}{2}K\varepsilon^{-d}|A| + D\varepsilon^{-d}|A|\right),$$

and by using (16), one has that (17) can be further estimated as

$$\begin{aligned} & \exp \left(-\frac{K\varepsilon^{-d}}{N-2} - \frac{1}{2}K\varepsilon^{-d}|A| + D\varepsilon^{-d}|A| \right) + \int_{\mathcal{V}(u,A,\kappa,\varepsilon)} \exp(-H(\varphi, A, \varepsilon)) \\ & \geq \exp \left(-\frac{K\varepsilon^{-d}}{N-2} \right) \sum_{i=1}^N \int_{\mathcal{N}_i} \exp(-H(\varphi, A_i, \varepsilon) - H(\varphi, A \setminus \bar{A}_i)) . \end{aligned}$$

We also notice that by using (18) one has that

$$\sum_{i=1}^N \int_{\mathcal{N}_i} \exp(H(\varphi, A_i, \varepsilon) - H(\varphi, A \setminus \bar{A}_i, \varepsilon)) \geq \int_{\mathcal{V}(u,A,\varepsilon)} \exp(-H(\varphi, A, \varepsilon)) ,$$

thus there exists $1 \leq i_0 \leq N$ such that

$$\int_{\mathcal{N}_{i_0}} \exp(-H(\varphi, A_{i_0}, \varepsilon) - H(\varphi, A \setminus \bar{A}_{i_0}, \varepsilon)) \geq \frac{1}{N} \int_{\mathcal{V}(u,A,\varepsilon)} \exp(-H(\varphi, A, \varepsilon)) . \quad (19)$$

Without loss of generality, we may assume that $i_0 = 1$. Hence, combining (19) with the previous estimates we have that

$$\begin{aligned} & \exp \left(-\frac{K}{N-2} - \frac{1}{2}K\varepsilon^{-d}|A| + D\varepsilon^{-d}|A| \right) + \int_{\mathcal{V}(u,A,\kappa,\varepsilon)} \exp(-H(\varphi, A, \varepsilon)) \\ & \geq \frac{1}{N} \exp \left(-\frac{K}{N-2} \right) \int_{\mathcal{N}_1} \exp(-H(\varphi, A_1, \varepsilon) - H(\varphi, A \setminus \bar{A}_1)) . \end{aligned}$$

We notice that the variables $H(\varphi, A_1, \varepsilon)$ and $H(\varphi, A \setminus \bar{A}_1, \varepsilon)$ are “independent”, thus by using the Fubini theorem one has that

$$\begin{aligned} & \int_{\mathcal{V}(u,A,\kappa,\varepsilon)} \exp(-H(\varphi, A_1, \varepsilon) - H(\varphi, A \setminus \bar{A}_1)) \geq \int_{\mathcal{V}(u,A_1,\kappa,\varepsilon)} \exp(-H(\varphi, A_1, \varepsilon)) \\ & \quad \times \int_{\mathcal{V}(u,A \setminus \bar{A}_1,\kappa,\varepsilon)} \exp(-H(\varphi, A \setminus \bar{A}_1)) , \end{aligned}$$

where in the previous inequality we have also used that

$$\mathcal{V}(u, A \setminus \bar{A}_1, \kappa, \varepsilon) \cap \mathcal{V}(u, A_1, \kappa, \varepsilon) \subset \mathcal{V}(u, A, \kappa, \varepsilon) .$$

To summarize, we have proved that for A_1

$$\begin{aligned} & \varepsilon^{-d} \log \left(\exp(F(u, A, \kappa, \varepsilon)) + \exp \left(-\frac{K}{N-2} - \frac{1}{2}K\varepsilon^{-d}|A| + D\varepsilon^{-d}|A| \right) \right) \\ & \leq -\varepsilon^d \log \left(\frac{K}{N-2} \right) + \varepsilon^d \log(N) + F(u, A_1, \kappa, \varepsilon) + F(u, A \setminus \bar{A}_1, \kappa, \varepsilon) . \end{aligned}$$

Finally, to conclude it is enough to pass to the limit in ε , then in N and then in κ , and use the “almost” monotonicity of the map $A \mapsto F''(u, A)$ (see Remark 2.10) and Lemma 2.11 to estimate the term $F(u, A \setminus \bar{A}_1, \kappa, \varepsilon)$. \square

Lemma 2.14. *Let A be an open set with piecewise regular regular boundary, suppose that ∂A has finite length, and let $u \in W^{1,p}(\mathbb{R}^d)$. Then the followings hold*

$$F'(u, A) = F'_\infty(u, A) \quad \text{and} \quad F''(u, A) = F''_\infty(u, A).$$

Proof. Without loss of generality, we may assume that $u = 0$. Indeed, if it is possible to change the boundary condition to 0 it is possible to change the boundary condition for every $u \in W^{1,p}(A)$ as this would correspond to a translation of the field by $\varphi_{u,\varepsilon}$ in all the formulas, hence leaving the integrals unchanged.

Because $H_\infty(\varphi, A, \varepsilon) \geq H(\varphi, A, \varepsilon)$, it is not difficult to notice that

$$F'_\infty(u, A, \kappa, \varepsilon) \geq F'(u, A, \kappa, \varepsilon) \quad \text{and} \quad F''_\infty(u, A, \kappa, \varepsilon) \geq F''(u, A, \kappa, \varepsilon).$$

The rest of the proof will consist in proving the opposite.

Let $A' \subset\subset A$ and a family of sets $(A_i)_{i=1}^n$ so that

$$A_i := \{x \in A_\varepsilon : \text{dist}(x, A^C) > t_i\} \quad \text{and} \quad S_i^{\xi, \varepsilon} := \{x \in A_i : x + \varepsilon \xi \notin A_i\}$$

here $t_i = i \frac{\text{dist}(A', A^C)}{n}$. From Lemma 2.6, one has that for every δ_0 , there exists $N \in \mathbb{N}$ such that

$$\sum_{|\xi| > N} \sum_{x \in A_\varepsilon} f_{\xi, \varepsilon}(x, \nabla_\xi \varphi) \leq \sum_{|\xi| > N} \sum_{x \in A_\varepsilon} C_\xi |\nabla \varphi(x)|^p \leq \delta_0 \sum_{x \in A_\varepsilon} |\nabla \varphi(x)|^p.$$

Moreover, one has that

$$\sum_{\xi \in \mathbb{Z}^d} \sum_{x \in A_\varepsilon} f_{\xi, \varepsilon}(x, \nabla \varphi(x)) \leq \sum_{|\xi| < N} \sum_{x \in A_\varepsilon} f_{\xi, \varepsilon}(x, \nabla \varphi(x)) + \delta_0 \sum_{x \in A_\varepsilon} |\nabla \varphi(x)|^p.$$

The right hand side can be rewritten by

$$\begin{aligned} & \sum_{|\xi| < N} \sum_{x \in A_i} f_{\xi, \varepsilon}(x, \nabla \varphi(x)) + \sum_{|\xi| < N} \sum_{x \in S_i^{\xi, \varepsilon}} f_{\xi, \varepsilon}(x, \nabla \varphi(x)) + \sum_{|\xi| < N} \sum_{x \in A_\varepsilon \setminus A_i} f_{\xi, \varepsilon}(x, \nabla \varphi(x)) \\ & + \delta_0 \sum_{x \in A_\varepsilon} |\nabla \varphi(x)|^p. \end{aligned}$$

On the other side, because

$$\sum_{i=1}^n \sum_{|\xi| < N} \sum_{x \in S_i^{\xi, \varepsilon}} f_{\xi, \varepsilon}(x, \nabla \varphi(x)) \leq C_N H(\varphi, A, \varepsilon),$$

there exists i such that

$$\sum_{x \in S_i^{\xi, \varepsilon}} f_{\xi, \varepsilon}(x, \nabla \varphi(x)) \leq \frac{C_N}{n} H(\varphi, A, \varepsilon). \tag{20}$$

Because of the above, the range of interactions N will be fixed, and take n large. Combining the above we have

$$H_\infty(\varphi, A, \varepsilon) \leq H(\varphi, A_i, \varepsilon) + \frac{C_N}{n} H_\infty(\varphi, A, \varepsilon) + H_\infty(\varphi, A \setminus A_i, \varepsilon)$$

As in the proof of Lemma 2.13, by using Lemma 2.12 one can show that there exists $i \in \{1, \dots, n\}$ and K (depending eventually on N, δ_0, p), such that $\limsup_{\varepsilon \rightarrow 0} F_\infty(0, A, \kappa, \varepsilon)$ can be bounded from above by

$$\limsup -\varepsilon^d \log \int_{\mathcal{V}(0, A, \kappa)} \exp \left(-H(\varphi, A_i, \varepsilon) - H_\infty(\varphi, A \setminus A_i, \varepsilon) - K/n\varepsilon^{-d} \right).$$

Finally, the claim can be obtained by using the regularity of $F'(u, \cdot)$ and $F''(u, \cdot)$, and the fact that $F''(u, C) \rightarrow 0$ as $|C| \rightarrow 0$.

□

Lemma 2.15 (subadditivity). *Let $A', A, B', B \subset \Omega$ be open sets such that $A' \Subset A$ and such that $B' \Subset B$. Then for every $u \in W^{1,p}$ one has that*

$$F''(u, A' \cup B') \leq F''(u, A) + F''(u, B).$$

Proof. The proof of this statement is very similar to Lemma 2.13.

□

Lemma 2.16 (locality). *Let $u, v \in W^{1,p}(\Omega)$ such that $u \equiv v$ in A . Then*

$$F'(u, A) = F'(v, A) \quad \text{and} \quad F''(u, A) = F''(v, A) \quad (21)$$

Proof. The statement follows from the definitions.

□

Proof of Theorem 2.5.

Let us suppose initially that there exists a sequence for which $F(\cdot, \cdot) = F'(\cdot, \cdot) = F''(\cdot, \cdot)$. Then to conclude it is enough to notice that F satisfies the conditions of Theorem 2.2. Indeed, in the previous Lemmas we prove that all the conditions (i)-(v) of Theorem 2.2 hold.

□

Corollary 2.17. *Because of Lemma 2.14, the same statement holds true for F_∞ . This in particular implies that for the sequence $\{\varepsilon_{n_k}\}$ in Theorem 2.5 there holds a large deviation principle with rate functional*

$$I(v) = \int_{\Omega} W(x, \nabla v) dx - \min_{\bar{v} \in W_0^{1,p}(\Omega) + u} \int_{\Omega} W(x, \nabla \bar{v}(x)) dx. \quad (22)$$

2.4 Homogenisation

In this section we will show that if the functions $f_{\xi, \varepsilon}$ are obtained by rescaling by ε in the space variable, then a LDP result holds true. This models the case when the arrangement of the “material points” presents a periodic feature, namely:

(H1) periodicity:

$$f_{\xi,\varepsilon}(x,t) = f^\xi\left(\frac{x}{\varepsilon}, t\right)$$

where the functions f^ξ are such that $f^\xi(x + Me_i, t) = f^\xi(x, t)$.

(H2) lower bound on the nearest neighbours:

$$f^{e_i}(x, t) \geq c_1(|t|^p - 1)$$

(H3) upper bound

$$f^\xi(x, t) \leq C_\xi(|t|^p + 1)$$

The main objective of this section is to prove the following homogenization result:

Theorem 2.18. *Let the functions $f_{\xi,\varepsilon}^\xi$ satisfy the above conditions. Then there exists a function f_{hom} such that for every $A \subset \Omega$ open set it holds*

$$F(u, A) = \begin{cases} \int_A f_{\text{hom}}(\nabla u) & \text{if } u \in W^{1,p}(\Omega; \mathbb{R}^d) \\ +\infty & \text{otherwise,} \end{cases} \quad (23)$$

where

$$f_{\text{hom}}(M) := \frac{1}{|A|} \lim_{\varepsilon \downarrow 0} F'(Mx, A, \kappa, \varepsilon). \quad (24)$$

Proof. Let (ε_n) be a sequence of positive numbers converging to 0. From Proposition 2.5 we can extract a subsequence (that we do not relabel for simplicity) such that

$$F'_{\{\varepsilon_n\}}(u, A) = F''_{\{\varepsilon_n\}}(u, A) = \int_A f_{\{\varepsilon_n\}}(x, \nabla u) \, dx.$$

The theorem is proved if we show that f does not depend on the space variable x and on the chosen sequence ε_n . To prove the first claim, by Theorem 2.2, it suffices to show that, if one denotes by

$$F(u, A) = \int_A f(x, \nabla u) \, dx,$$

then

$$F(Mx, B(y, \rho)) = F(Mx, B(z, \rho))$$

for all $M \in \mathbb{R}^{d \times m}$, $y, z \in \Omega$ and $\rho > 0$ such that $B(y, \rho) \cup B(z, \rho) \subset \Omega$. We will prove that

$$F(Mx, B(y, \rho)) \leq F(Mx, B(z, \rho)).$$

The proof of the opposite inequality is analogous.

Let $x, y \in \mathbb{R}^d$ and let $x_\varepsilon = \arg \min(\text{dist}(y, x + (\varepsilon M)\mathbb{Z}^d))$. Then $x_\varepsilon \rightarrow y$ as $\varepsilon \downarrow 0$. From the periodicity hypothesis, one has that

$$F(M, B(x, \rho, \kappa, \varepsilon)) = F(M, B(x_\varepsilon, \rho, \kappa, \varepsilon)) \leq F(M, B(y, \rho + \delta, \kappa, \varepsilon))$$

where in the last inequality we have used the monotonicity with respect to the inclusion relation of $A \mapsto F(u, A, \kappa, \varepsilon)$ and δ is such that $|y - x_\varepsilon| \leq \delta$.

Let us now turn to the independence on the sequence on the chosen sequence. Let us initially notice that because of the LDP, whenever $u = Mx$ where M is a linear map it holds

$$F'(u, A, \kappa) = F'(u, A) \quad \text{and} \quad F''(u, A, \kappa) = F''(u, A). \quad (25)$$

Because of Theorem 2.2, it is enough to show that for every linear map M the following limit exists and

$$\frac{1}{|A|} \lim_{\varepsilon \downarrow 0} F'(Mx, A, \kappa, \varepsilon)$$

The existence of the above limit (and its independence on κ) follows easily by the standard methods with the help of an approximative subadditivity. A simple proof can be found in [8, Proposition 1.2]. \square

3 SBV Representation Theorem

In this section we extend the results of the previous section to more general local interactions, where the problem relaxes naturally in SBV. The strategy will be very similar to the one used in § 2. However, we will need to use different tools and a different Representation Theorem. Repeating many of the arguments in the previous section is thus unavoidable, however we will refer to the previous section often, when the repetition becomes pedantic.

3.1 A *very* short introduction to SBV

Before going into the details of our main Theorem of this section, let us define the functional spaces BV and SBV. For a general introduction on these spaces see [2]. However, please notice that the definitions given in this section differ slightly from the ones in [2]. More precisely, in the following, we additionally impose the finiteness of $(n - 1)$ -Hausdorff measure of the jump set. This technical modification is done in order to have at our disposal general representation theorems like the ones in the following section.

Let Ω be an open set. We say that $u \in L^1(\Omega)$ belongs to $\text{BV}(\Omega)$, if there exists a vector measure $Du = (D_1u, \dots, D_nu)$ with finite total variation in Ω , such that

$$\int_{\Omega} u \partial_i \varphi \, dx = - \int \varphi \, dD_i u \quad \forall \varphi \in C_0^1(\Omega)$$

Let $Du = D^a u + D^s u$ be the Radon-Nikodym decomposition of Du in absolutely continuous and singular part with respect to the \mathcal{L}^n and let ∇u be the density of $D^a u$. It can be seen that u is approximately differentiable at x and the approximate differential equals to $\nabla u(x)$, i.e.,

$$\lim_{\rho \downarrow 0} \rho^{-n} \int_{B_\rho(x)} \frac{|u(y) - u(x) - \langle \nabla u, y - x \rangle|}{|y - x|} dy = 0$$

for \mathcal{L}^n -a.e. $x \in \Omega$.

For the singular part, it is useful to introduce the upper and lower approximate limits u_+, u_- , defined by

$$\begin{aligned} u_-(x) &= \inf \{t \in [-\infty, +\infty] : \{x \in \Omega : u(x) > t\} \text{ has density 0 at } x\} \\ u_+(x) &= \sup \{t \in [-\infty, +\infty] : \{x \in \Omega : u(x) < t\} \text{ has density 0 at } x\}. \end{aligned}$$

It is well-known that $u_+(x) \in \mathbb{R}$ for \mathcal{H}^{d-1} -a.e. $x \in \Omega$. The jump set S_u is defined by

$$S_u := \{x \in \Omega : u_-(x) < u_+(x)\}.$$

We define the jump part Ju of the derivative as the restriction of $D^s u$ to the jump set S_u . We also recall that there exists a Borel map $\nu_u : S_u \rightarrow S^{d-1}$ such that

$$\nu_{E_t}(x) = \nu_u \quad \text{for } \mathcal{H}^{d-1} \text{ a.e. } x \in \partial^* E_t \cap S_u$$

for any t such that $E_t := \{x : u > t\}$.

Proposition 3.1. *Let $u \in BV(\Omega)$. Then, the jump part of the derivative is absolutely continuous with respect to \mathcal{H}^{d-1} and*

$$Ju = (u_+ - u_-)\nu_u \mathcal{H}^{d-1} \llcorner S_u$$

Finally, we define the space $SBV_p(\Omega)$ as the set of functions $u \in BV(\Omega)$ such that $\nabla u \in L^p(\Omega)$, $D^s u = Ju$ and

$$\mathcal{H}^{d-1}(S_u) < +\infty. \tag{26}$$

Note that in [2] the condition (26) is not imposed.

3.2 Preliminary results

Let us now recall some well-known results, which will be useful in the sequel.

Theorem 3.2 ([2, Theorem 4.7]). *Let $\varphi : [0, +\infty) \rightarrow [0, +\infty]$, $\theta : [0, +\infty) \rightarrow [0, +\infty]$ be a lower semicontinuous function increasing functions and assume that*

$$\lim_{t \rightarrow +\infty} \frac{\varphi(t)}{t} = +\infty \quad \text{and} \quad \lim_{t \rightarrow 0} \frac{\theta(t)}{t} = +\infty \tag{27}$$

Let $\Omega \subset \mathbb{R}^n$ be an open and bounded and let $(u_h) \subset SBV(\Omega)$ such that

$$\sup \left\{ \int_{\Omega} \varphi(|\nabla u_h|) + \int_{J_{u_h}} \theta(|u_h^+ - u_h^-|) d\mathcal{H}^{n-1} \right\} < +\infty. \quad (28)$$

If (u_h) weakly* converges in $BV(\Omega)$, then $u \in SBV(\Omega)$, the approximate gradients ∇u_h weakly converge to $\nabla u \in (L^1(\Omega))^N$. $D_j u_h$ weakly* converge to $D_j u \in \Omega$ and

$$\begin{aligned} \int_{\Omega} \varphi(|\nabla u|) dx &\leq \liminf_{h \rightarrow +\infty} \int_{\Omega} \varphi(|\nabla u_h|) dx \quad \text{if } \varphi \text{ is convex} \\ \int_{J_u} \theta(|u^+ - u^-|) d\mathcal{H}^{n-1} &\leq \liminf_{h \rightarrow +\infty} \int_{J_{u_h}} \theta(|u_h^+ - u_h^-|) d\mathcal{H}^{n-1} \end{aligned}$$

if θ is concave.

Theorem 3.3 (Compactness SBV [2, Theorem 4.8]). *Let φ, θ as in Theorem 3.2. Let (u_h) in $SBV(\Omega)$ satisfy (28) and assume in addition that $\|u_h\|_{\infty}$ is uniformly bounded in h . Then there exists a subsequence (u_{h_k}) weakly* converging in $BV(\Omega)$ to $u \in SBV(\Omega)$.*

We now give the a set of condition which give a representation formula similar to the one of Theorem 2.2.

Let

$$\mathcal{F} : SBV_p(\Omega, \mathbb{R}^d) \times \mathcal{A}(\Omega) \rightarrow [0, +\infty]$$

such that the followings hold:

- (H1) $\mathcal{F}(u, \cdot)$ is the restriction to $\mathcal{A}(\Omega)$ of a Radon measure,
- (H2) $\mathcal{F}(u, A) = \mathcal{F}(v, A)$ whenever $u = v$ \mathcal{L}^n a.e. on $A \in \mathcal{A}(\Omega)$,
- (H3) $\mathcal{F}(\cdot, A)$ is L^1 l.s.c.,
- (H4) there exists a constant C such that

$$\begin{aligned} &\frac{1}{C} \left(\int_A |\nabla u|^p dx + \int_{S(u) \cap A} (1 + |u^+ - u^-|) d\mathcal{H}^{n-1} \right) \\ &\leq \mathcal{F}(u, A) \\ &\leq C \left(\int_A |\nabla u|^p dx + \int_{S(u) \cap A} (1 + |u^+ - u^-|) d\mathcal{H}^{n-1} \right). \end{aligned} \quad (29)$$

Here, Ω is an open bounded set of \mathbb{R}^n . As before, $\mathcal{A}(\Omega)$ is the class of all open subsets of Ω and $SBV_p(\Omega)$ is the space of functions $u \in SBV(\Omega)$ such that $\nabla u \in L^p(\Omega)$ and $\mathcal{H}^{n-1}(J_u) < +\infty$. For every $u \in SBV_p(\Omega)$ and $A \in \mathcal{A}(\Omega)$ define

$$m(u; A) := \inf \{ \mathcal{F}(w; A) : w \in SBV_p(\Omega) \text{ such that } w = u \text{ in a neighbourhood of } \partial A \}$$

The role of Theorem 2.2, will be played by the following result, whose proof can be founded in [3].

Theorem 3.4. *Under hypotheses (H1)-(H4), for every $u \in SBV_p(\Omega)$ and $A \in \mathcal{A}(\Omega)$ there exists a function W_1 and W_2 such that W_1 is quasi-convex, W_2 is BV-elliptic and such that*

$$\mathcal{F}(u, A) := \int_A W_1(x, u, \nabla u) dx + \int_{A \cap S_u} W_2(x, u^+, u^-, \nu_u) d\mathcal{H}^{n-1}.$$

Moreover, the functions W_1 and W_2 can be computed via

$$W_1(x_0, u_0, \cdot) := \limsup_{\varepsilon \rightarrow 0^+} \frac{\mathbf{m}(u_0 + \xi(\cdot - x_0), Q(x_0, \varepsilon))}{\varepsilon^n}$$

$$W_2(x_0, a, b, \nu) := \limsup_{\varepsilon \rightarrow 0^+} \frac{\mathbf{m}(u_{x_0, a, b, \nu}, Q_\nu(x_0, \varepsilon))}{\varepsilon^{n-1}}$$

for all $x_0 \in \Omega$, $u_0, a, b \in \mathbb{R}^d$, $\xi \in \mathbb{R}^d$, $\nu \in S^{n-1}$ and where

$$u_{x_0, a, b, \nu}(x) := \begin{cases} a & \text{if } (x - x_0) \cdot \nu > 0, \\ b & \text{if } (x - x_0) \cdot \nu \leq 0. \end{cases}$$

As $u_{x_0, b, a, \nu} = u_{x_0, a, b, \nu}$ \mathcal{L}^n a.e. in $Q_\nu(x_0, \varepsilon) = Q_{-\nu}(x_0, \varepsilon)$, one has that

$$W_2(x_0, b, a, -\nu) = W_2(x_0, a, b, \nu),$$

for every $x_0 \in \Omega$, $a, b \in \mathbb{R}^d$ and $\nu \in \mathbb{R}^d$.

Remark 3.5. Condition (29), can be softened to

$$\begin{aligned} & \frac{1}{C} \left(\int_A |\nabla u|^p dx + \int_{S(u) \cap A} (|u^+ - u^-|) d\mathcal{H}^{n-1} \right) \\ & \leq \mathcal{F}(u, A) \\ & \leq C \left(\int_A |\nabla u|^p dx + \int_{S(u) \cap A} (|u^+ - u^-|) d\mathcal{H}^{n-1} \right). \end{aligned} \tag{30}$$

Indeed, let us suppose that \mathcal{F} satisfies only (30). By the same theorem (Theorem 3.4) it is possible to represent $\mathcal{F} \text{cal}(u, A) + \mathcal{H}(S_u \cap A)$, thus by removing the subtracted part it is possible to represent \mathcal{F} .

Finally, let us recall also the following result:

Theorem 3.6 ([7]). *Assume that $\partial\Omega$ is locally Lipschitz and let $u \in SBV_p(\Omega, \mathbb{R}^m)$. for every $\varepsilon > 0$ there exists a function $v \in SBV_p(\Omega, \mathbb{R}^n)$ such that*

- (i) S_v is essentially closed.
- (ii) $\overline{S_v}$ is a polyhedral set
- (iii) $\|u - v\|_{L^p} \leq \varepsilon$
- (iv) $\|\nabla u - \nabla v\| \leq \varepsilon$
- (v) $|\mathcal{H}^{n-1}(S_u) - \mathcal{H}^{n-1}(S_v)| \leq \varepsilon$
- (vi) $v \in C^\infty(\Omega \setminus \overline{S_v})$

3.3 Hypothesis and Main Theorem

Given Theorem 3.2, it is natural to impose the following hypothesis.

Let $g^{(1)}$ a monotone convex functions such that there exists a constant C such that

$$g^{(1)}(t) \geq C \max(t^p - 1, 0)$$

and $g^{(2)}$ be a monotone concave function such that

$$g^{(2)}(t) \geq c > 0 \quad \text{and} \quad \lim_{t \uparrow \infty} \frac{g^{(1)}(t)}{t} = +\infty.$$

The typical example we have in mind is when $g^{(1)}(t) := t^p$ and $g^{(2)}(t) := 1 + t^\alpha$, where $0 < \alpha < 1$ and $p > 1$.

Let $T_\varepsilon \uparrow \infty$ be such that $\varepsilon T_\varepsilon \downarrow 0$. We denote

$$g_\varepsilon(x) = \begin{cases} g^{(1)}(\|x\|) & \text{if } \|x\| < T_\varepsilon, \\ \frac{1}{\varepsilon} g^{(2)}(\varepsilon \|x\|) & \text{if } \|x\| \geq T_\varepsilon. \end{cases}$$

We will also assume that there exists a constant C such that $g^{(1)}(T_\varepsilon) \leq \frac{C}{\varepsilon} g^{(2)}(T_\varepsilon \varepsilon)$, and that for every $M > 0$ there exists a constant C_M such that

$$g_\varepsilon(M|t|) \leq C_M g_\varepsilon(|t|).$$

Let $(f_{\xi, \varepsilon})$ be a family of local interactions such that for every ξ, ε it holds

$$f_{\xi, \varepsilon}(x, t) \lesssim C_\xi (g_\varepsilon(|t|) + 1) \tag{31}$$

with $\sum_{\xi \in \mathbb{Z}^d} |\xi| C_\xi < +\infty$, and such that for every $1 \leq j \leq d$ it holds

$$f_{e_i, \varepsilon}(x, t) \gtrsim (g_\varepsilon(|t|) - 1) \tag{32}$$

We will assume also that there exists a constant $M < +\infty$ such that

$$\int_{\mathbb{R}} \exp(-g_\varepsilon(t)) \, dt \leq M.$$

Let us now define the Hamiltonians as

$$H(u, A, \varepsilon) = \sum_{\xi \in \mathbb{Z}^N} \sum_{x \in R_\varepsilon^\xi(A)} f_{\xi, \varepsilon}\left(x, \frac{\varphi(x + \varepsilon \xi) - \varphi(x)}{|\xi|}\right)$$

and

$$H_\infty(\varphi, A, \varepsilon) := \sum_{\xi \in \mathbb{Z}^d} \sum_{x \in A_\varepsilon} f_{\varepsilon, \xi}(x, \nabla_\xi \varphi(x)).$$

Remark 3.7. Let $u \in SBV_p(\Omega) \cap L^\infty(\Omega)$. Then one can show there exists an discretized $\varphi_{u,\varepsilon}$ such that

$$\|u\|_{SBV_p} \lesssim \varepsilon^d \sum_{x \in \varepsilon \mathbb{Z}^d \cap \Omega} g_\varepsilon(\nabla \varphi_{u,\varepsilon}) \lesssim \|u\|_{SBV_p}.$$

Indeed, whenever u is piecewise in C^∞ , the statement is trivial. In order to conclude the general case it is enough to use Theorem 3.6.

Let us discuss very informally the above hypothesis. The function g_ε will play the role of $\|\cdot\|^p$ in § 2 and the conditions on $g^{(1)}$ and $g^{(2)}$ are in order to ensure the compactness and lower semicontinuity. Given that a discrete function can be interpolated by continuous functions, it does not make sense to talk about jump set. However, it makes sense to consider as a jump set, the set of points where the discrete gradient is bigger than a certain threshold T_ε . Indeed, if we were approximating a function with a jump, it is expected that the gradient would explode (in a neighbourhood of the jump set) like δ/ε , where δ is the amplitude of the jump and ε is the discretization parameter. Thus $T_\varepsilon \uparrow \infty$. Indeed, suppose that the function we are approximating is $\delta \chi_B$, where δ is a small parameter and B is the unit ball. Then the jump set would be the set of points where the gradient goes like $\frac{\delta}{\varepsilon}$. Thus in order to “catch” jumps of order δ one needs that the $\lim_{\varepsilon \downarrow 0} T_\varepsilon \varepsilon \leq \delta$. Thus $\lim_{\varepsilon \downarrow 0} T_\varepsilon \varepsilon = 0$.

As in the previous section, one of the main steps will be to show that $F'_\infty = F'$ and that $F''_\infty = F''$. The basic intuition behind, is again a version of the interpolation lemma. As before, we will show that if one imposes “closeness” v in $L^p(A)$ to some regular function u , then one can impose also the boundary condition by “paying a very small price in energy”. More precisely, given a sequence $\{v_n\}$ such that $v_n \rightarrow u$ in $L^p(A)$, where A is an open set, then there exists a sequence $\{\tilde{v}_n\}$ such that $\tilde{v}_n \rightarrow u$, such that $\tilde{v}_n|_{\partial\Omega} = u|_{\partial\Omega}$ and

$$\liminf_n \|\tilde{v}_n\|_{SBV_p(A)} \leq \liminf_n \|v_n\|_{SBV_p(A)}.$$

Remark 3.8. Let $f : [0, +\infty) \rightarrow [0, +\infty)$ be a monotone function. Then, it is immediate to have

$$f\left(\frac{1}{N} \sum_{i=1}^N t_i\right) \leq \sum_{i=1}^N f(t_i),$$

where $t_i > 0$.

Similarly as in § 2, for every $A \in \mathcal{A}(\Omega)$, we define the free-energy as

$$F(u, A, \kappa, \varepsilon) := -\varepsilon^d \log \int_{\mathcal{V}(u, A, \kappa)} \exp\left(-H(\varphi, A, \varepsilon)\right) d\varphi$$

$$F_\infty(u, A, \kappa, \varepsilon) := -\varepsilon^d \log \int_{\mathcal{V}_\infty(u, A, \kappa)} \exp\left(-H_\infty(\varphi, A, \varepsilon)\right) d\varphi$$

where

$$\mathcal{V}(u, A, \kappa) = \left\{ \varphi : A_\varepsilon \rightarrow \mathbb{R}^m \mid \frac{\varepsilon^d}{|A|^d} \sum_{x \in A_\varepsilon} |u - \varepsilon \varphi|^p \leq \kappa^p \right\}$$

$$\mathcal{V}_\infty(u, A, \kappa) = \left\{ \varphi : \varepsilon \mathbb{Z}^d \rightarrow \mathbb{R}^m \mid \frac{\varepsilon^d}{|A|^d} \sum_{x \in A_\varepsilon} |u - \varepsilon \varphi|^p \leq \kappa^p, \text{ and } \varphi(x) = \varphi_{u,\varepsilon}(x) \ \forall x \notin A_\varepsilon \right\},$$

where $\varphi_{u,\varepsilon}$ is defined in (1).

Similarly as in § 2, let us introduce the following notations:

$$\begin{aligned}
F'(u, A, \kappa) &:= \liminf_{\varepsilon \downarrow 0} F(u, A, \kappa, \varepsilon) \\
F''(u, A, \kappa) &:= \limsup_{\varepsilon \downarrow 0} F(u, A, \kappa, \varepsilon) \\
F'(u, A) &:= \lim_{\kappa \downarrow 0} \liminf_{\varepsilon \downarrow 0} F(u, A, \kappa, \varepsilon) = \lim_{\kappa \downarrow 0} F'(u, A, \kappa) \\
F''(u, A) &:= \lim_{\kappa \downarrow 0} \limsup_{\varepsilon \downarrow 0} F(u, A, \kappa, \varepsilon) = \lim_{\kappa \downarrow 0} F''(u, A, \kappa) \\
F'_\infty(u, A, \kappa) &:= \liminf_{\varepsilon \downarrow 0} F_\infty(u, A, \kappa, \varepsilon) \\
F''_\infty(u, A, \kappa) &:= \limsup_{\varepsilon \downarrow 0} F_\infty(u, A, \kappa, \varepsilon) \\
F'_\infty(u, A) &:= \lim_{\kappa \downarrow 0} \liminf_{\varepsilon \downarrow 0} F_\infty(u, A, \kappa, \varepsilon) = \lim_{\kappa \downarrow 0} F'_\infty(u, A, \kappa) \\
F''_\infty(u, A) &:= \lim_{\kappa \downarrow 0} \limsup_{\varepsilon \downarrow 0} F_\infty(u, A, \kappa, \varepsilon) = \lim_{\kappa \downarrow 0} F''_\infty(u, A, \kappa)
\end{aligned}$$

We are now able to write the main result of this section.

Theorem 3.9. *Assume the previous hypothesis and that $u \in SBV_p \cap L^\infty$. Then for every infinitesimal sequence (ε_n) there exists a subsequence ε_{n_k} and functions $W_1 : \Omega \times \mathbb{R}^{d \times m} \rightarrow \mathbb{R}$ and $W_2 : \Omega \times \mathbb{R}^m \times S^{d-1} \rightarrow \mathbb{R}$ such that*

$$F(u, A) := F'_{n_k}(u, A) = F''_{n_k}(u, A) = \int_A W_1(x, \nabla u) dx + \int_{S_u} W_2(x, u^+(x) - u^-(x), \nu_u(x)),$$

where the function W_1 is a quasiconvex function and W_2 is a BV-elliptic function and depend on the chosen subsequence $\{\varepsilon_{n_k}\}$.

3.4 Proofs

The next technical lemma is a version of Lemma 2.6, that asserts that finite difference quotients along any direction can be controlled by finite difference quotients along the coordinate directions.

Lemma 3.10. *Let $A \subset \mathcal{A}(\Omega)$ and set $A_\varepsilon = \{x \in A : \text{dist}(x, A) > 2\sqrt{N}\varepsilon\}$. Then there exists a dimensional constant $C := C(N)$ such that for any $\xi \in \mathbb{Z}^N$ there holds*

$$\sum_{x \in R_\varepsilon^{e_i}(A_\varepsilon)} g_\varepsilon(\nabla_\xi u(x)) \leq C|\xi| \sum_{i=1}^N \sum_{x \in R_\varepsilon^{e_i}(A)} g_\varepsilon(\nabla_{e_i} u(x)).$$

Proof. As in the proof of Lemma 2.6, let $\xi \in \mathbb{Z}^d$. By decomposing it into coordinates, it is not difficult to notice that it can be written as

$$\xi = \sum_{k=1}^{N_\xi} \alpha_k(\xi) e_{i_k},$$

where $N_\xi \leq \delta|\xi|$ and $\alpha_k(\xi) \in \{-1, 1\}$. Denote by

$$\xi_k = \sum_{j=1}^{N_\xi} \alpha_k(\xi),$$

hence $|\xi_k| \leq |\xi|$ for all k . Thus

$$\nabla_\xi u(x) = \frac{1}{|\xi|} \sum_{k=1}^{N_\xi} \nabla_{\alpha_k(\xi)e_i} u(x + \varepsilon \xi_k)$$

Moreover, by the monotonicity of g_ε , we have

$$g_\varepsilon \left(\frac{1}{N_\xi} \sum_{k=1}^{N_\xi} \nabla_{\alpha_k(\xi)e_i} u(x + \varepsilon \xi_k) \right) \leq \sum_{k=1}^{N_\xi} g_\varepsilon \left(\nabla_{\alpha_k(\xi)e_i} u(x + \varepsilon \xi_k) \right)$$

Finally by summing over all ξ , exchanging the sums and using the equivalence of the norms i.e., $|\xi| \leq N_\xi \leq d|\xi|$ one has the desired result. □

As in the previous section, let G^λ be the free-energy (see (4) for the definition) induced by the Hamiltonian

$$\tilde{H}^\lambda(\varphi, A, \varepsilon) := \lambda \sum_{i=1}^d \sum_{x \in R_\varepsilon^{e_i}(A)} g_\varepsilon(|\nabla_i \varphi|).$$

In a very similar fashion as in Lemma 2.8, one can prove

Lemma 3.11. *There exists constants C_λ, D_λ , such that it holds*

$$C_\lambda \leq G^\lambda(0, A, \kappa, \varepsilon) \leq D_\lambda$$

The next proof is the analog of Lemma 2.9.

Lemma 3.12. *Let $\{f_{\xi, \varepsilon}\}$ satisfy the usual hypothesis. Then there exists a constant $D > 0$ and $\varepsilon_0 > 0$ such that for every $\kappa < 1$ it holds*

$$\exp \left(-\varepsilon^{-d} F(u, A, \kappa, \varepsilon) \right) \leq \exp \left(D \varepsilon^{-d} + D \sum_{\xi \in R_\varepsilon^{e_i}(A)} \sum_{i=1}^d g_\varepsilon(\nabla_{e_i} \varphi_{u, \varepsilon}) \right), \quad (33)$$

where $\varphi_{u, \varepsilon}$ is defined in (1).

Proof. Given that $g_\varepsilon(|a|) \lesssim g_\varepsilon(|a-b|) + g_\varepsilon(|b|)$ one has that there exist constants C_1 such that

$$\begin{aligned} H(\varphi, A, \varepsilon) &\geq C_1 \sum_{i=1}^d \sum_{x \in R_\varepsilon^{e_i}(A)} g_\varepsilon(|\nabla_{e_i} \varphi(x)|) \\ &\geq C_1 \sum_{i=1}^d \sum_{x \in R_\varepsilon^{e_i}(A)} g_\varepsilon(|\nabla \psi|) - \tilde{C}_1 \sum_{i=1}^d \sum_{x \in R_\varepsilon^{e_i}(A)} g_\varepsilon(|\nabla_{e_i} \varphi_{u,\varepsilon}(x)|) \end{aligned}$$

where $\psi = \varphi - \varphi_{u,\varepsilon}$. Hence the estimate (33) reduces to prove that there exists a constant D such that

$$\int_{\{\|\varepsilon\varphi\| \leq \kappa\}} \exp \left(-C \sum_{i=1}^d \sum_{x \in R_\varepsilon^{e_i}(A)} g_\varepsilon(|\nabla_{e_i} \varphi|) \right) \leq \exp(D\varepsilon^{-d}).$$

The above follows from Lemma 3.11. □

As in Remark 2.10, we have the following:

Remark 3.13. *Let $u \in L^\infty \cap SBV_p$, then along the lines of Lemma 3.12 one can easily prove that there exists a constant C such that*

$$A \mapsto F'(u, A) + C(|u|_{SBV_p(A)} + 1) \quad A \mapsto F''(u, A) + C(|u|_{SBV_p(A)} + 1)$$

are monotone with respect to the inclusion relation.

Lemma 3.14. *Let $f_{\xi,\varepsilon}$ satisfy our hypothesis and let A be an open set. Then there exists a constant $C, D > 0$ such that*

$$\exp \left(-\varepsilon^{-d} F(u, A, \kappa, \varepsilon) \right) \geq \exp \left(-D\varepsilon^{-d} - C \sum_{i=1}^d \sum_{x \in R_\varepsilon^{e_i}} g_\varepsilon(|\nabla_{e_i} \varphi_{u,\varepsilon}(x)|) \right)$$

where $\varphi_{u,\varepsilon}$ is defined in (1).

Proof. Using Lemma 3.10, one has that there exists a constant C such that

$$H(\varphi, A, \varepsilon) \leq C \sum_{i=1}^d \sum_{x \in R_\varepsilon^{e_i}(A)} g_\varepsilon(|\nabla_{e_i} \varphi|)$$

Given that $g_\varepsilon(a+b) \leq g_\varepsilon(2a) + g_\varepsilon(2b) \lesssim g_\varepsilon(a) + g_\varepsilon(b)$, there exist a constant C_1 such that

$$H(\varphi, A, \varepsilon) \leq C_1 \sum_{i=1}^d \sum_{x \in R_\varepsilon^{e_i}(A)} (g_\varepsilon(|\nabla_{e_i} \varphi_{\varepsilon,u}|) + 1) + 2d \sum_{x \in A_\varepsilon} g_\varepsilon(|\psi(x)|),$$

where $\psi = \varphi - \varphi_{u,\varepsilon}$. Hence, the estimate (33) reduces to prove that there exists a constant D such that

$$\int_{\mathcal{V}(0,A,\kappa,\varepsilon)} \exp \left(-C \sum_{x \in A_\varepsilon} g_\varepsilon(|\psi(x)|) \right) \geq (\varepsilon\kappa)^{-d} \exp \left(D\varepsilon^{-d} \right).$$

The above inequality was proved in Lemma 2.8. □

Lemma 3.15 (exponential tightness). *Let A be an open set and $K \geq 0$. Denote by*

$$\mathcal{M}_K := \left\{ \varphi : H(\varphi, A, \varepsilon) \geq K\varepsilon^{-d}|A| \right\}.$$

Then there exists a constant D, K_0, ε_0 such that for every $K \geq K_0$, $\varepsilon \leq \varepsilon_0$ it holds

$$\int_{\mathcal{M}_K \cap \mathcal{V}(u,A,\kappa)} \exp \left(-H(\varphi, A, \varepsilon) \right) \leq \exp \left(-\frac{1}{2}K\varepsilon^{-d} + D\varepsilon^{-d} - D \sum_{i=1}^d \sum_{x \in R_\varepsilon^{e_i}(A)} g_\varepsilon(|\nabla_{e_i} \varphi_u|) \right)$$

Proof. For every $\varphi \in \mathcal{M}_K$ it holds

$$H(\varphi, A, \varepsilon) \geq K/2\varepsilon^{-d} + H(\varphi, A, \varepsilon).$$

Hence, by using Lemma 3.14 we have the desired result. □

The proof of the following lemma is similar to Lemma 2.13.

Lemma 3.16 (regularity). *Let f_ξ satisfy the usual hypothesis and $u \in SBV_p \cap L^\infty$ then*

$$\sup_{A' \Subset A} F''(u, A') = F''(u, A).$$

Proof. Let us fix $A' \Subset A$ and $N \in \mathbb{N}$ (to be chosen later). Let $\delta = \text{dist}(A', A^C)$, and let $0 < t_1, \dots, t_N \leq \delta$ such that $t_{i+1} - t_i > \frac{\delta}{2N}$. Without loss of generality, we may assume that there exists no $x \in A_\varepsilon$ such that $\text{dist}(x, A^C) = t_i$. For every i we define

$$A_i := \{x \in A_\varepsilon : \text{dist}(x, A^C) \geq t_i\}$$

and

$$S_i^{\xi, \varepsilon} := \{x \in (A_i)_\varepsilon : x + \varepsilon\xi \in A \setminus A_i\}.$$

We have that

$$R_\varepsilon^\xi(A) = R_\varepsilon^\xi(A') + R_\varepsilon^\xi(A \setminus \bar{A}') + S_i^{\varepsilon, \xi}$$

thus

$$H^\xi(\varphi, A, \varepsilon) = H^\xi(\varphi, A \setminus \bar{A}_i, \varepsilon) + H^\xi(\varphi, A_i, \varepsilon) + \sum_{x \in S_i^{\xi, \varepsilon}} f_{\xi, \varepsilon}(\nabla_\xi \varphi(x)).$$

Hence,

$$H(\varphi, A, \varepsilon) = H(\varphi, A_i, \varepsilon) + H(\varphi, A \setminus A_i, \varepsilon) + \sum_{\xi \in \mathbb{Z}^d} \sum_{x \in S_i^{\xi, \varepsilon}} C_\xi \left(g_\varepsilon(|\nabla_\xi \varphi(x)|) + 1 \right)$$

Let us now estimate the last term in the previous inequality.

We separate the sum into two terms

$$\sum_{\xi \in \mathbb{Z}^d} \sum_{x \in S_i^{\xi, \varepsilon}} g_\varepsilon(|\nabla_\xi \varphi(x)|) = \sum_{|\xi| \leq M} \sum_{x \in S_i^{\xi, \varepsilon}} g_\varepsilon(|\nabla_\xi \varphi(x)|) + \sum_{|\xi| > M} \sum_{x \in S_i^{\xi, \varepsilon}} g_\varepsilon(|\nabla_\xi \varphi(x)|). \quad (34)$$

Let $M \in \mathbb{N}$. From the condition (31) and by taking M sufficiently large, we may also assume without loss of generality that

$$\sum_{|\xi| \geq M} |\xi| C_\xi \leq \delta_1.$$

Hence, by using Lemma 3.10 we have that

$$\sum_{|\xi| \geq M} \sum_{x \in S_i^{\xi, \varepsilon}} g_\varepsilon(|\nabla_\xi \varphi(x)|) \leq C \delta_1 \sum_{k=1}^d \sum_{x \in R_{\varepsilon^k}(A)} g_\varepsilon(|\nabla_{e_k} \varphi(x)|) \leq \tilde{C} \delta_1 H(\varphi, A, \varepsilon),$$

where in the last inequality we have used hypothesis (32).

Let $|\xi| < M$. If $\varepsilon M N \leq 2\delta$, then for every

$$S_i^{\xi, \varepsilon} \cap S_j^{\xi, \varepsilon} = \emptyset \quad \text{whenever } |i - j| \geq 2.$$

Without loss of generality, we may assume the above condition as $\varepsilon \rightarrow 0$.

Given that

$$\frac{1}{N-2} \sum_{i=1}^{N-2} \sum_{|\xi| < M} \sum_{x \in S_i^{\varepsilon, \xi}} g_\varepsilon(|\nabla_\xi \varphi(x)|) \leq 2CH(\varphi, A, \varepsilon)$$

there exist $0 < i \leq N-2$ such that

$$\sum_{|\xi| < M} \sum_{x \in S_i^{\xi, \varepsilon}} g_\varepsilon(|\nabla_\xi \varphi|) < \frac{2}{N-2} H(\varphi, A, \varepsilon). \quad (35)$$

Let us denote by \mathcal{N}_i the set of all $\varphi \in \mathcal{V}(u, A, \kappa, \varepsilon)$ such that (35) holds for the first time, namely for every $j \leq i$

$$\sum_{|\xi| < M} \sum_{x \in S_i^{\xi, \varepsilon}} g_\varepsilon(|\nabla_\xi \varphi|) \geq \frac{2}{N-2} H(\varphi, A, \varepsilon). \quad (36)$$

On one side, we have that

$$\int_{\mathcal{V}(u, A, \kappa, \varepsilon)} \exp(-H(\varphi, A, \varepsilon)) \leq \sum_{i=1}^N \int_{\mathcal{N}_i} \exp(-H(\varphi, A_i, \varepsilon) - H(\varphi, A \setminus \bar{A}_i, \varepsilon)),$$

on the other side one has that

$$\int_{\mathcal{V}(u, A, \kappa, \varepsilon)} \exp(-H(\varphi, A, \varepsilon)) \geq \sum_{i=1}^N \int_{\mathcal{N}_i^K} \exp(-H(\varphi, A, \varepsilon)),$$

where $\mathcal{N}_i^K := \mathcal{N}_i \setminus \mathcal{M}_K$. By using (36), one has that for every $\varphi \in \mathcal{N}_i^K$ it holds

$$H(\varphi, A, \varepsilon) \leq H(\varphi, A_i) + H(\varphi, A \setminus \bar{A}_i) + \frac{K\varepsilon^{-d}}{N-2}$$

and for every φ it holds

$$H(\varphi, A, \varepsilon) \geq H(\varphi, A, \varepsilon) + H(\varphi, A \setminus \bar{A}_i, \varepsilon).$$

Hence,

$$\int_{\mathcal{V}(u, A, \kappa, \varepsilon)} \exp(-H(\varphi, A, \varepsilon)) \geq \sum_{i=1}^N \int_{\mathcal{N}_i^K} \exp\left(-H(\varphi, A_i) - H(\varphi, A \setminus \bar{A}_i) - \frac{K\varepsilon^{-d}}{N-2}\right).$$

From now on the proof follows as in Lemma 2.13. □

Lemma 3.17. *For every open set A and $u \in SBV_p(\mathbb{R}^d) \cap L^\infty$ it holds*

$$F'(u, A) = F'_\infty(u, A) \quad \text{and} \quad F'(u, A) = F'_\infty(u, A).$$

Proof. The proof of the above statement follows in the same way as in Lemma 2.14. □

Lemma 3.18 (subadditivity). *Let $A', A, B', B \subset \Omega$ be open sets such that $A' \Subset A$ and such that $B' \Subset B$. Then for every $u \in SBV_p \cap L^\infty$ one has that*

$$F''(u, A' \cup B') \leq F''(u, A) + F''(u, B)$$

Proof. The proof of this statement is very similar to Lemma 3.16 and Lemma 3.17. □

Lemma 3.19 (locality). *Let $u, v \in SBV_p(\Omega) \cap L^\infty$ such that $u \equiv v$ in A . Then*

$$F'(u, A) = F'(v, A) \quad \text{and} \quad F''(u, A) = F''(v, A)$$

Proof. The statement follows from the definitions. □

Proof of Theorem 3.9. Let us suppose initially that there exists a sequence for which $F(\cdot, \cdot) = F'(\cdot, \cdot) = F''(\cdot, \cdot)$. Then to conclude it is enough to notice that F satisfies the conditions of Theorem 3.4, which are proved in the previous Lemmas. □

Corollary 3.20. *Because of Lemma 3.17, the same statement holds true for F_∞ . This in particular implies that for the sequence $\{\varepsilon_{n_k}\}$ in Theorem 3.9 there holds a large deviation principle with rate functional*

$$I(v) = \int_{\Omega} W_1(x, \nabla v) dx \int_{J_u} W_2(x, u_+(x) - u_-(x)) d\mathcal{H}^{d-1}(x) - \min_{\bar{v} \in W_0^{1,p}(\Omega)+u} \int_{\Omega} W(\nabla \bar{v}(x)) dx. \quad (37)$$

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